Formalizing an optimization problem for inferring the entropy production from observed statistics

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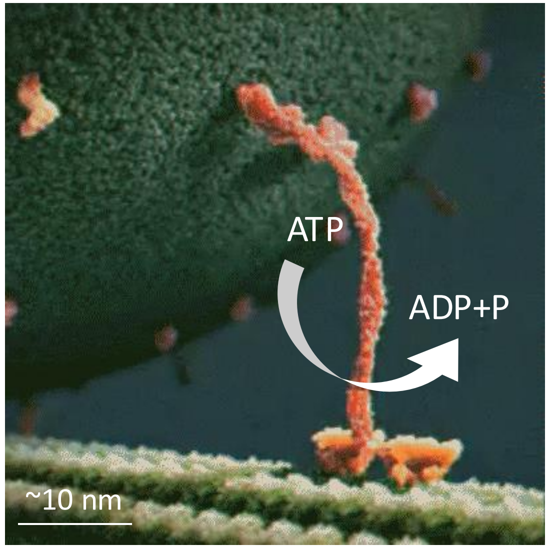
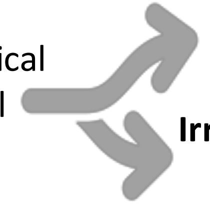
TEL AVIV UNIVERSITY

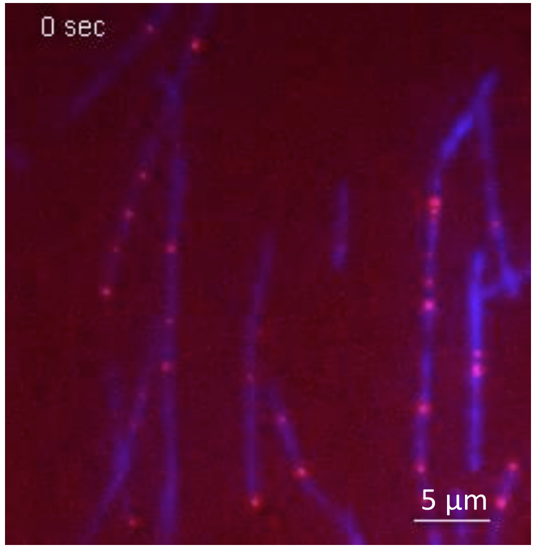
Quantitative irreversibility



- Small systems are subjected to thermal fluctuations.



- Entropy production constrains the amount of extractable work
- Only partial information is available.



The entropy production rate $\dot{S}$ can be estimated by the distinguishability between forward and reverse trajectories, quantified by the KLD.

Entropy production rate

Distinguishability between forward and reverse trajectories

$$\dot{S} \geq \dot{S}_{\mathrm{KLD}} \equiv \lim_{t\to\infty} \frac{k_B}{t} D[\mathcal{P}(\gamma_t)||\mathcal{P}(\tilde{\gamma}_t)]$$

Forward trajectory of duration t

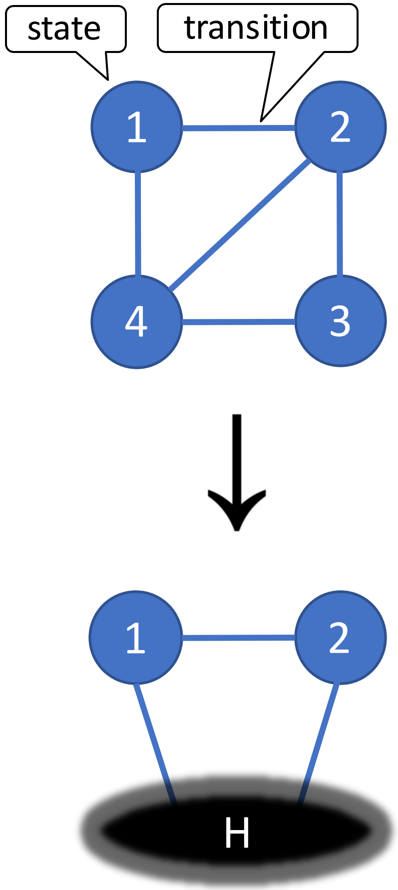
Reverse trajectory

Parrondo, den Broeck, Kawai, New J. Phys., 2009

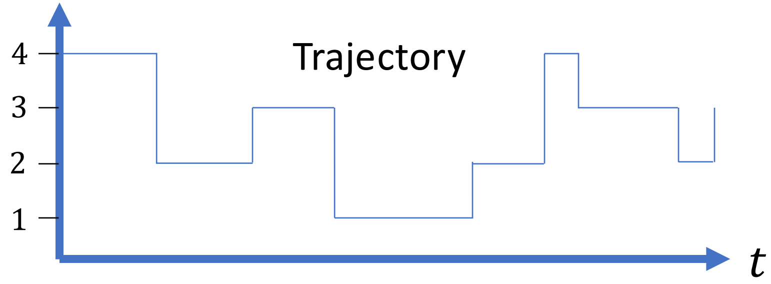
The Kullback-Leibler divergence (KLD), or the relative entropy, measures of how different two probability distributions are.

$$D[p(x)||q(x)] = \int_x dx\, p(x) \ln \frac{p(x)}{q(x)} \geq 0$$

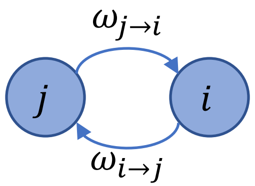
Stochastic Thermodynamics



Continuous-Time Markov Chain (CTMC)



- W – Rate matrix
- $W_{ij} = \omega_{j\to i}$ transition rate $j \to i$

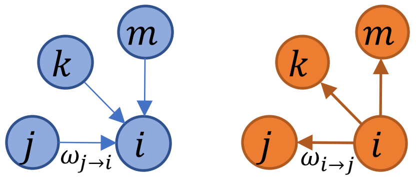


- Dynamics: $\dot{p} = Wp$

$$\dot{p}_i = \sum_j W_{ij} p_j = \sum_{j\neq i} \left(p_j \omega_{j\to i} - p_i \omega_{i\to j} \right)$$

At the long-time limit, the system reaches a **steady state**

$$\lim_{t\to\infty} p_i(t) = \pi_i$$



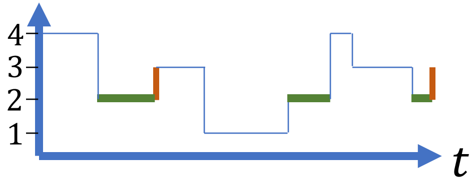
Total Entropy Production Rate (EPR)

$$\dot{S} = \sum_{j>i} \left(\pi_j \omega_{j\to i} - \pi_i \omega_{i\to j} \right) \ln \frac{\pi_j \omega_{j\to i}}{\pi_i \omega_{i\to j}} \geq 0$$

Current, $J_{j\to i}$ Thermodynamic force

Mass rates

$$n_{ij} = \pi_i \omega_{i\to j}$$

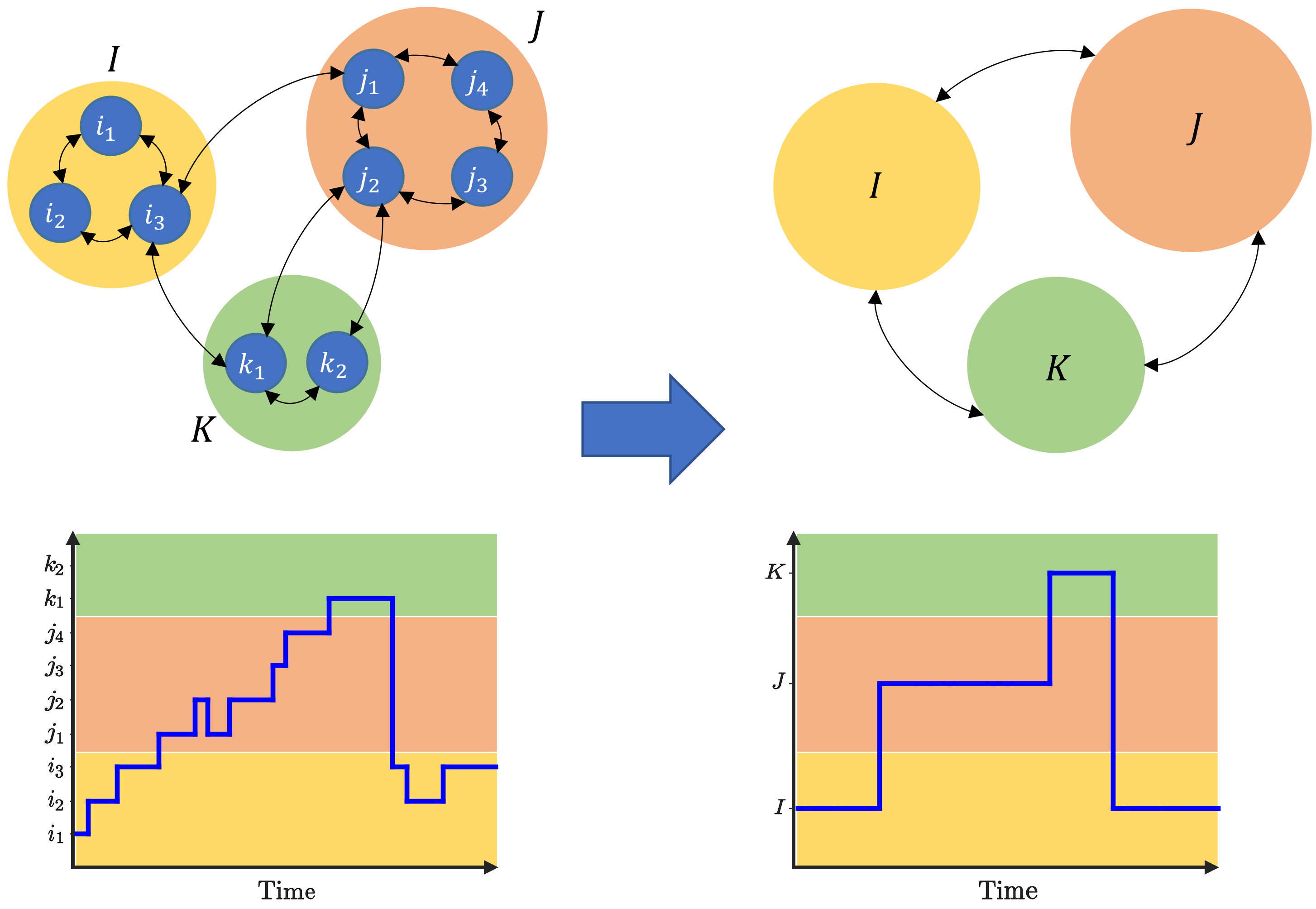


- Only partial information is available
- A lower bound on the total EPR
- Non-Markovian dynamics
  (2nd-order semi-Markov)

$$\dot{S} = \sum_{j>i} \left(n_{ji} - n_{ij} \right) \ln \frac{n_{ji}}{n_{ij}} \geq 0$$

$$\pi_2 = \frac{Time@2}{Total\ time} \qquad n_{23} = \frac{\#2 \to 3}{Total\ time}$$

Coarse-graining



How do we estimate the EPR from the observed trajectory?

Notice:
Non-Markovian dynamics

Eden Nitzan, Aishani Ghosal, and Gili Bisker, *Physical Review Research*, 5(4), 2023

Bounds on the total entropy production rate

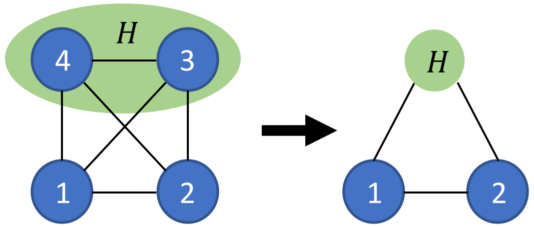
σ_{pp} - Passive Partial EPR

Shiraishi, N. and Sagawa, T. *Physical Review E*, 2015

- EPR from transitions between observed Markovian states

$$\sigma_{pp} = (\pi_1\omega_{1\to 2} - \pi_2\omega_{2\to 1}) \log\left(\frac{\pi_1\omega_{1\to 2}}{\pi_2\omega_{2\to 1}}\right)$$

$$= \underbrace{(n_{12} - n_{21})}_{\text{Current } 1 \to 2} \log\left(\frac{n_{12}}{n_{21}}\right)$$



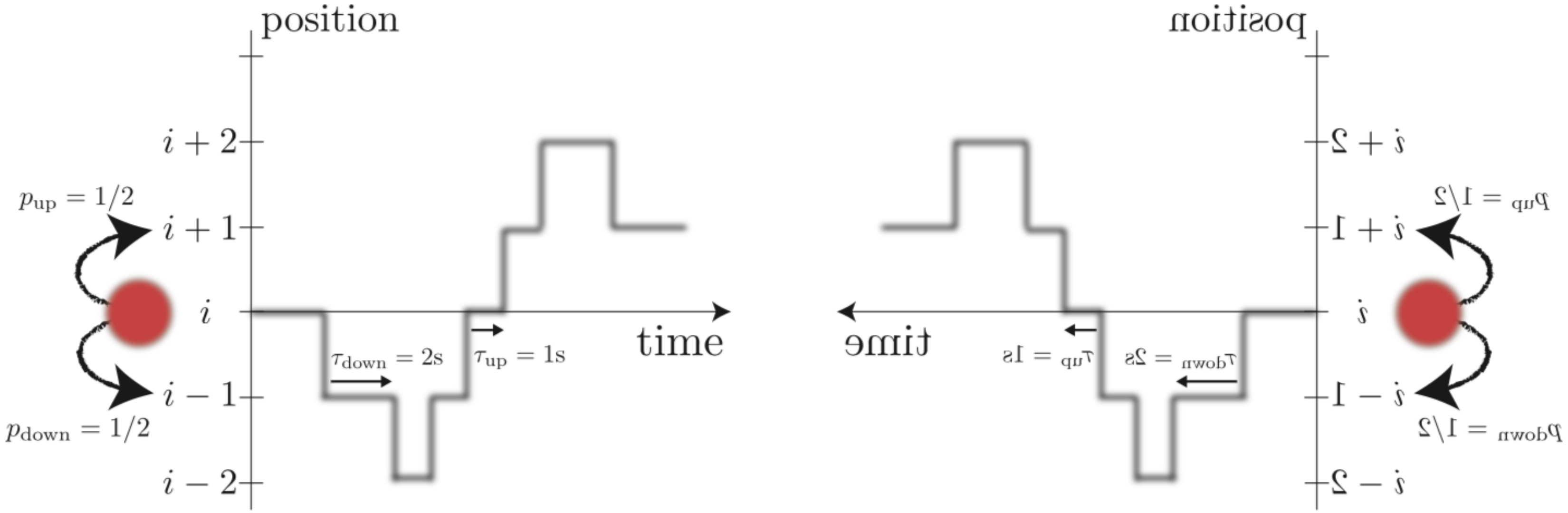
- No observed currents = trivial bound of 0

Bounds on the total entropy production rate



σ_{KLD} - Kullback–Leibler Divergence Bound

Ignacio A. Martínez*, Gili Bisker*, Jordan M. Horowitz, and Juan M.R. Parrondo, *Nature Communications*, 2019

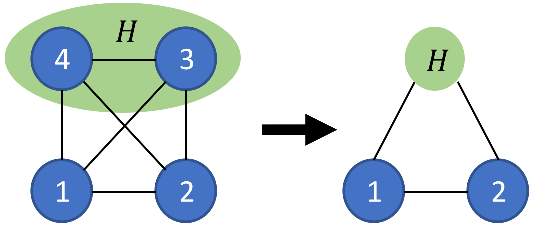


Note: this is a **non-Markovian** system, as the residence time depants on the direction of the transition

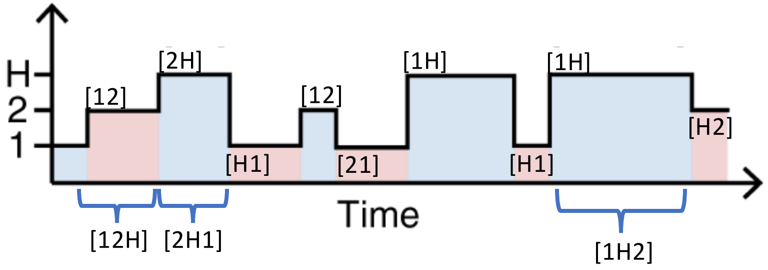
Bounds on the total entropy production rate

σ_{KLD} - Kullback–Leibler Divergence Bound

Ignacio A. Martínez*, Gili Bisker*, Jordan M. Horowitz, and Juan M.R. Parrondo, *Nature Communications*, 2019

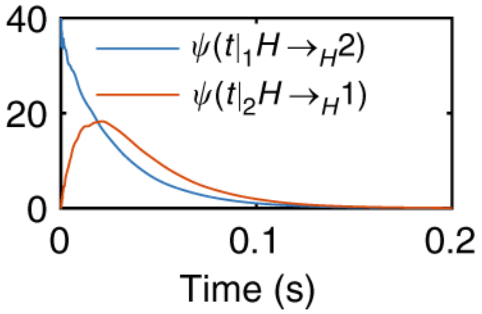


- Consider states $[ij]$
- Kullback-Leibler Divergence for second-order semi-Markov process
- $\sigma_{KLD} = \lim_{T\to\infty} \frac{1}{T} \mathcal{D}[\mathcal{P}(\gamma_T)||\mathcal{P}(\tilde{\gamma}_T)] = \sigma_{aff} + \sigma_{WTD}$



$$\sigma_{aff} = \frac{1}{\mathcal{T}} \sum_{I,J,K} p(IJK) \log\left(\frac{p([IJ] \to [JK])}{p([KJ] \to [JI])}\right)$$

$$\sigma_{WTD} = \frac{1}{\mathcal{T}} \sum_{I,J,K} p(IJK) \mathcal{D}[\psi(t|[IJ] \to [JK])||\psi(t|[KJ] \to [JI])]$$



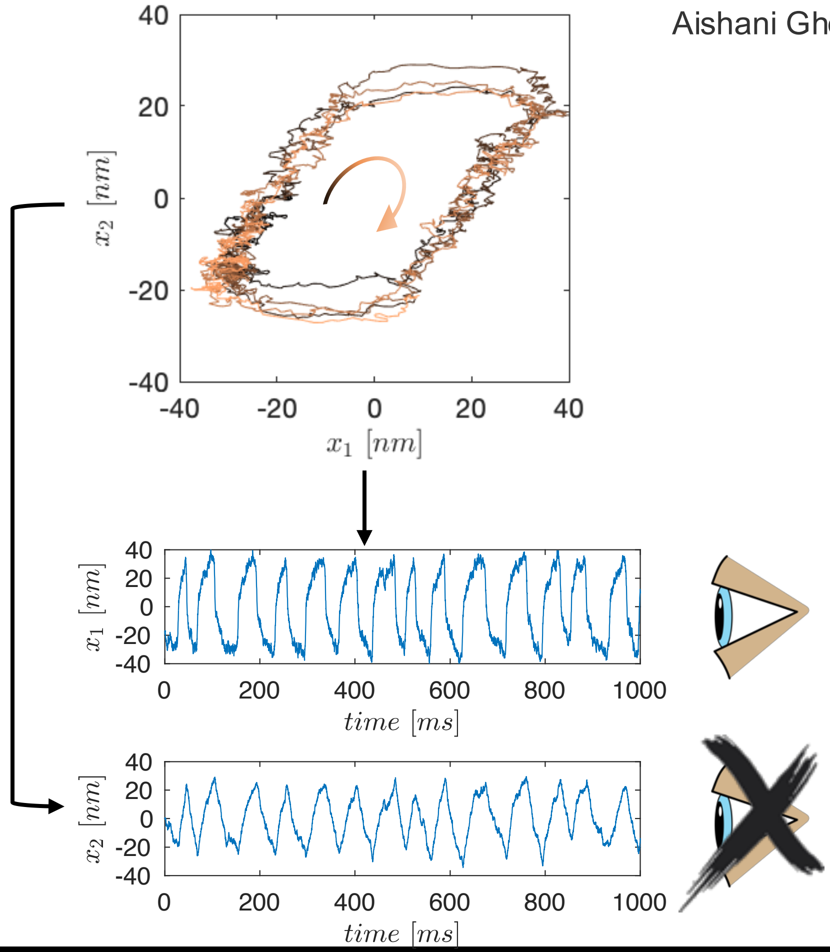
Bounds on the total entropy production rate

Inferring entropy production in continuous variable systems from partial information

Aishani Ghosal and Gili Bisker, *Physical Chemistry Chemical Physics* 24 (39), 2022

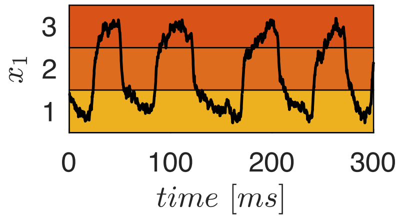
Dr. Aishani Ghosal

Current in $[x_1, x_2]$ plane

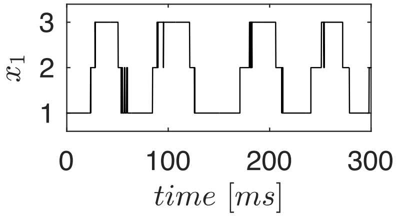


Observed

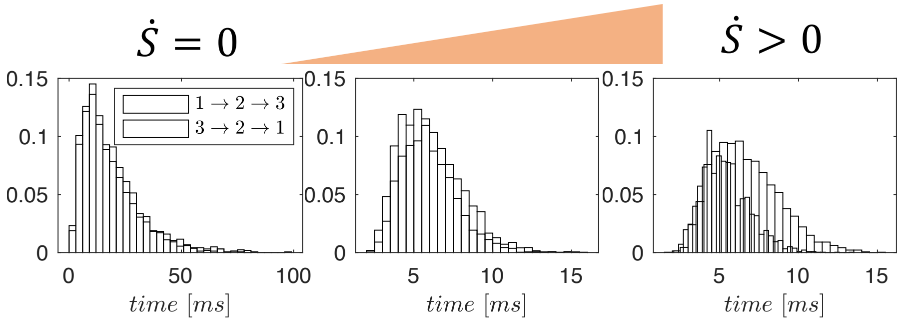
Unobserved



Coarse-graining



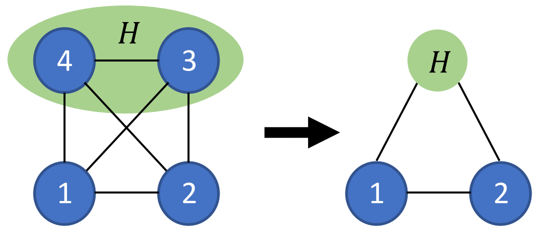
Currents may appear as only fluctuations in partially observed systems



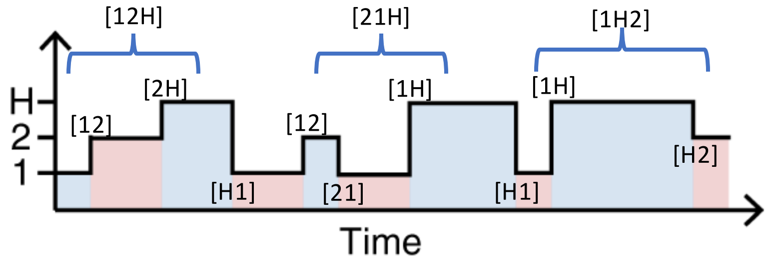
Bounds on the total entropy production rate

σ_2 Bound

Skinner, D. J. and Dunkel, J., *Proceedings of the National Academy of Sciences*, 2021



- Considering the first- ($\mathcal{O}_1$) and second-order ($\mathcal{O}_2$) transition statistics
  - $\mathcal{O}_1$ - first order observed mass rates n_{IJ}
  - $\mathcal{O}_2$ - second order observed mass rates n_{IJK}



Bounds on the total entropy production rate

σ_2 Bound

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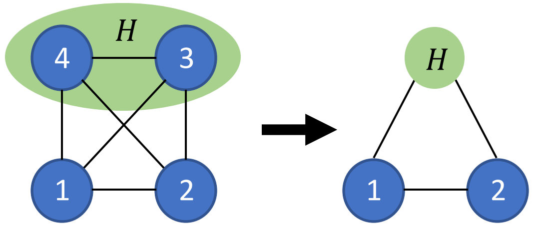
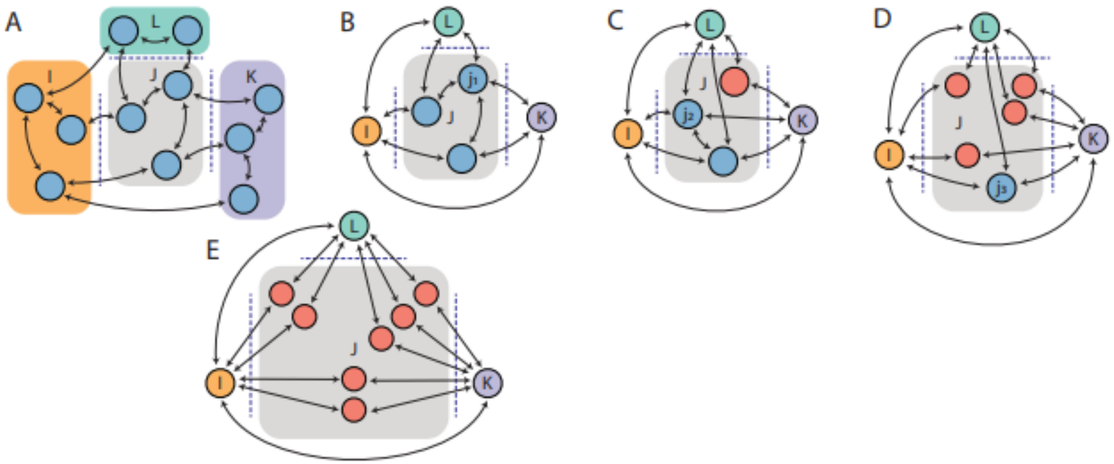
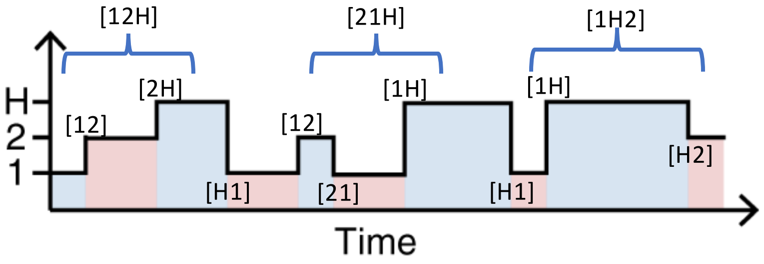


- Considering the first- ($\mathcal{O}_1$) and second-order ($\mathcal{O}_2$) transition statistics
- Transforming the system into a canonical form
  - Conserving the considered statistics
  - Not raising the EPR

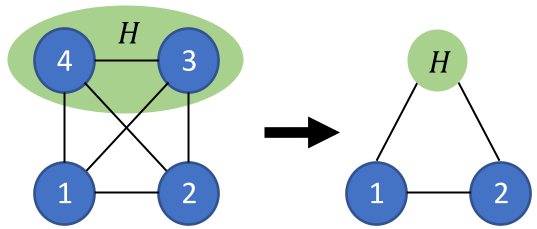




Bounds on the total entropy production rate

σ_2 Bound

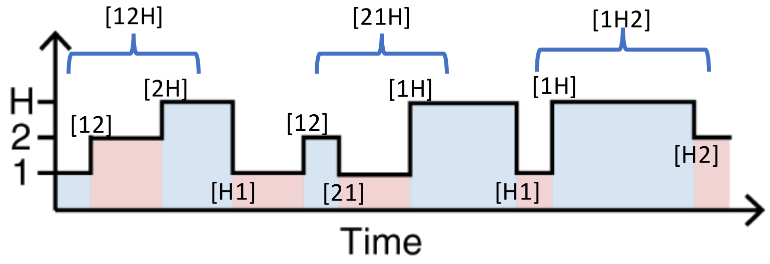
Skinner, D. J. and Dunkel, J., *Proceedings of the National Academy of Sciences*, 2021



- Considering the first- ($\mathcal{O}_1$) and second-order ($\mathcal{O}_2$) transition statistics
- Transforming the system into a canonical form
- Formalizing an optimization problem with constraints

$$\sigma_2(J) = \sum_{I \neq J \neq K} \min_{Canonical\ form\ \mathcal{R}_{IJK}} \{\sigma_{tot}(\mathcal{R}_{IJK}) | n_{IJ}, n_{JI}, n_{JK}, n_{KJ}, n_{IJK}, n_{KJI}\}$$

$$\sigma_2 = \frac{1}{2} \sum_J \sigma_2(J)$$



What information is exploited?

Bound	σ_{pp}	σ_{aff}	σ_{KLD}	σ_2	?
Information used	Transitions between observed Markovian states n_{ij}	Statistics for second-order semi-Markov process $[IJ] \to [JK]$ Transitions only!	Statistics for second-order semi-Markov process $[IJ] \to [JK]$ Transitions and waiting times	First- and second-order transitions statistics of the observed states n_{IJ}, n_{IJK}	

<u>Hierarchy of the bounds:</u>

$$\sigma_{pp} \overset{\color{blue}\bullet}{\leq} \sigma_{aff} \overset{\color{orange}\bullet}{\leq} \sigma_{KLD}$$

$$\sigma_{pp} \overset{\color{green}\bullet}{\leq} \sigma_2$$

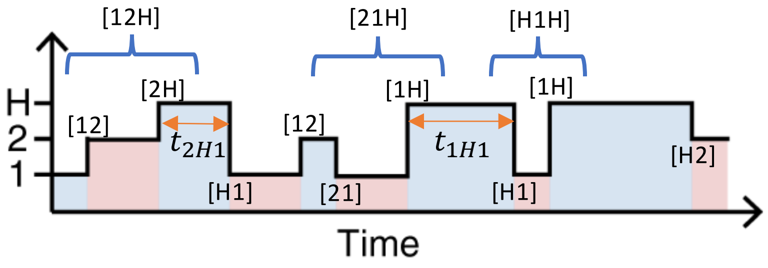
● Gili Bisker, Matteo Polettini, Todd R. Gingrich, and Jordan M. Horowitz, *J. of Statistical Mechanics*, 2017
● Ignacio A. Martínez*, Gili Bisker*, Jordan M. Horowitz, and Juan M.R. Parrondo, *Nature Communications*, 2019
● Skinner, D. J. and Dunkel, J., *Proceedings of the National Academy of Sciences*, 2021

Skinner & Dunkel:

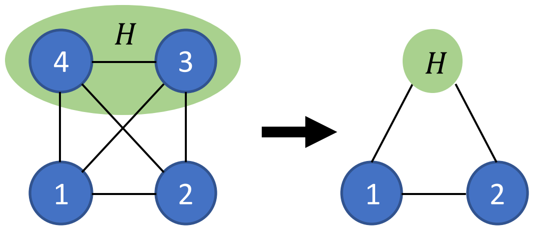
in addition to observing enough conditional transitions to construct both waiting time distributions. It is, in principle, possible to construct systems where $\sigma_{WTD} > \sigma_2$, although this does not apply for any of the synthetic and experimental data considered in the present study. In any case, if sufficient trajectories are observed to calculate waiting time distributions, one could define an estimator $\hat{\sigma} = \max\{\sigma_2, \sigma_1 + \sigma_{WTD}\}$, to construct what is currently the best possible estimate.

New bound on the total entropy production rate

σ_{opt}

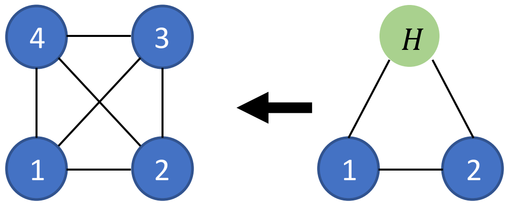


- Steady state probabilities – π_I
- First- and second-order transitions statistics – n_{IJ}, n_{IJK}
- Conditional waiting time statistics – $\psi_{IJK}(t)$
- Requirement: The underlying topology!



- $\sigma_{opt}^{(\infty)}(\mathcal{S}) = \min_{\mathcal{R}}\{\sigma_{tot}(\mathcal{R}) | \forall_{I,J,K}: \pi_I^{\mathcal{R}} = \pi_I^{\mathcal{S}}, n_{IJ}^{\mathcal{R}} = n_{IJ}^{\mathcal{S}}, n_{IJK}^{\mathcal{R}} = n_{IJK}^{\mathcal{S}}, \psi_{IJK}^{\mathcal{R}}(t) = \psi_{IJK}^{\mathcal{S}}(t)\}$

Eden Nitzan



Eden Nitzan, Aishani Ghosal, and Gili Bisker, *Physical Review Research*, 5(4), 2023

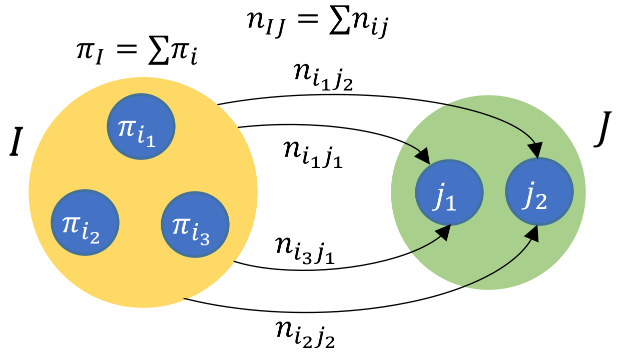
New bound on the total entropy production rate

σ_{opt}

The optimization variables

- Analytical expressions of the observables in terms of π_i, n_{ij}

$$\mathcal{R} = \mathcal{R}(\pi_i, n_{ij})$$



- Linear constraints:
    - $\pi_I = \sum_{i \in I} \pi_i$ - probability conservation
    - $n_{IJ} = \sum_{i \in I, j \in J} n_{ij}$ - mass conservation
    - $\forall_i : \sum_j n_{ij} = \sum_j n_{ji}$ - mass conservation at each state i

- Non-linear constraints:
    - $n_{IJK}(\pi_i, n_{ij})$
    - $\psi_{IJK}(t; \pi_i, n_{ij})$

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New bound on the total entropy production rate

σ_{opt}

- $\sigma_{opt}^{(\infty)}(\mathcal{S}) = \min_{\mathcal{R}}\{\sigma(\mathcal{R})|\forall_{I,J,K}\colon \pi_I^{\mathcal{R}} = \pi_I^{\mathcal{S}}, n_{IJ}^{\mathcal{R}} = n_{IJ}^{\mathcal{S}}, n_{IJK}^{\mathcal{R}} = n_{IJK}^{\mathcal{S}}, \psi_{IJK}^{\mathcal{R}}(t) = \psi_{IJK}^{\mathcal{S}}(t)\}$

- $\sigma_{opt}^{(n)}(\mathcal{S}) = \min_{\mathcal{R}}\left\{\sigma(\mathcal{R})|\forall_{I,J,K}\colon \pi_I^{\mathcal{R}} = \pi_I^{\mathcal{S}}, n_{IJ}^{\mathcal{R}} = n_{IJ}^{\mathcal{S}}, n_{IJK}^{\mathcal{R}} = n_{IJK}^{\mathcal{S}}, \forall_{k\in\{1,\ldots,n\}}\colon \langle t_{IJK}^k\rangle^{\mathcal{R}} = \langle t_{IJK}^k\rangle^{\mathcal{S}}\right\}$
- $\forall n \in N:\ \sigma_{tot}(\mathcal{S}) \geq \sigma_{opt}^{(\infty)}(\mathcal{S}) \geq \sigma_{opt}^{(n)}(\mathcal{S}) \geq \cdots \geq \sigma_{opt}^{(1)}(\mathcal{S})$

$$\langle t_{IJK}^k\rangle = (-1)^{\mathrm{k}} \frac{d^k \tilde{\psi}_{IJK}(s)}{ds^k}\bigg|_{s\to 0}$$

Eden Nitzan, Aishani Ghosal, and Gili Bisker, *Physical Review Research*, 5(4), 2023

New bound on the total entropy production rate

Bound	σ_{pp}	σ_{aff}	σ_{KLD}	σ_2	σ_{opt}
Information used	Transitions between observed Markovian states n_{ij}	Statistics for second-order semi-Markov process $[IJ] \rightarrow [JK]$ Transitions only!	Statistics for second-order semi-Markov process $[IJ] \rightarrow [JK]$ Transitions and waiting times	First- and second-order transitions statistics of the observed states n_{IJ}, n_{IJK}	Steady states probabilities π_I First- and second-order transitions statistics n_{IJ}, n_{IJK} Waiting time statistics $\langle t_{IJK}^k \rangle$ Underlying topology

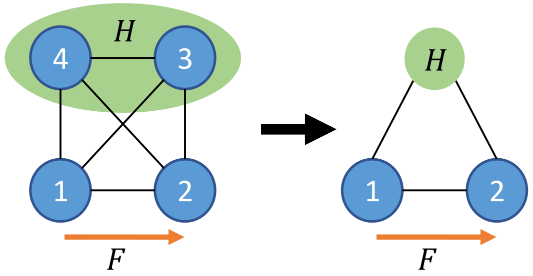
<u>Hierarchy of the bounds:</u>

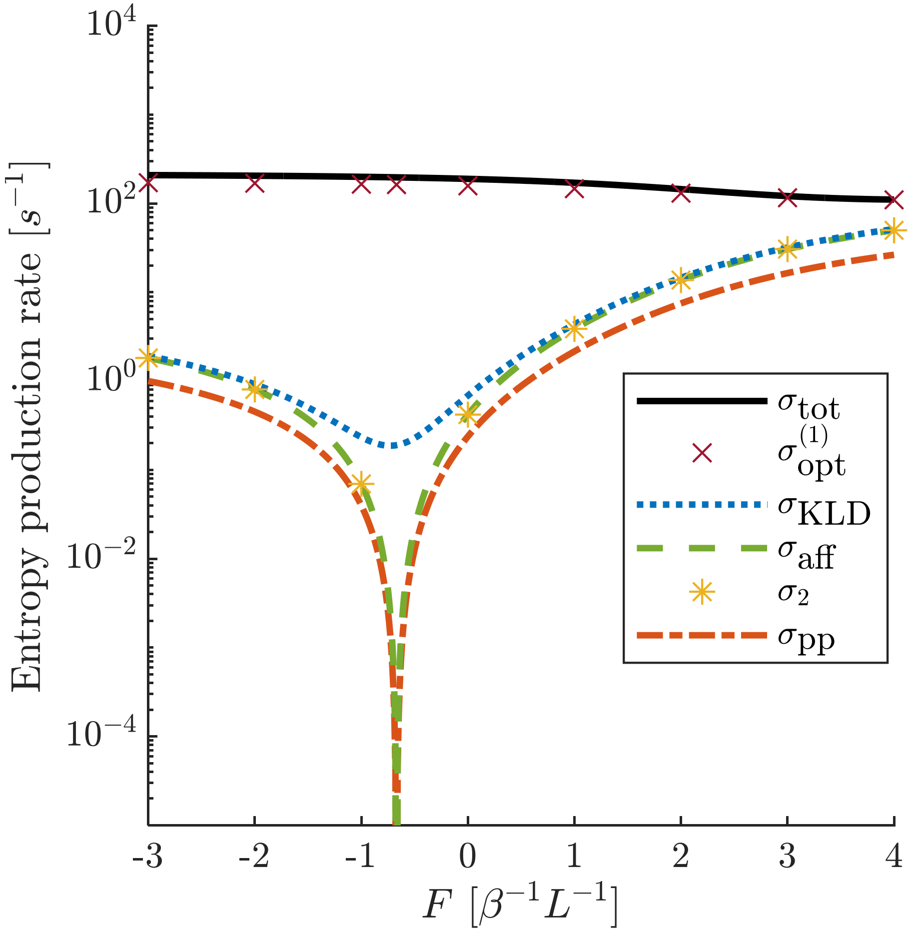
$$\left.\begin{array}{l} \sigma_{pp} \leq \sigma_{aff} \leq \sigma_{KLD} \\ \sigma_{pp} \leq \sigma_2 \end{array}\right\} \leq \sigma_{opt}$$

Gili Bisker, Matteo Polettini, Todd R. Gingrich, and Jordan M. Horowitz, *J. of Statistical Mechanics*, 2017
Ignacio A. Martínez*, Gili Bisker*, Jordan M. Horowitz, and Juan M.R. Parrondo, *Nature Communications*, 2019
Skinner, D. J. and Dunkel, J., *Proceedings of the National Academy of Sciences*, 2021
Eden Nitzan, Aishani Ghosal, and Gili Bisker, *Physical Review Research*, 2023

Results

- 4-state system





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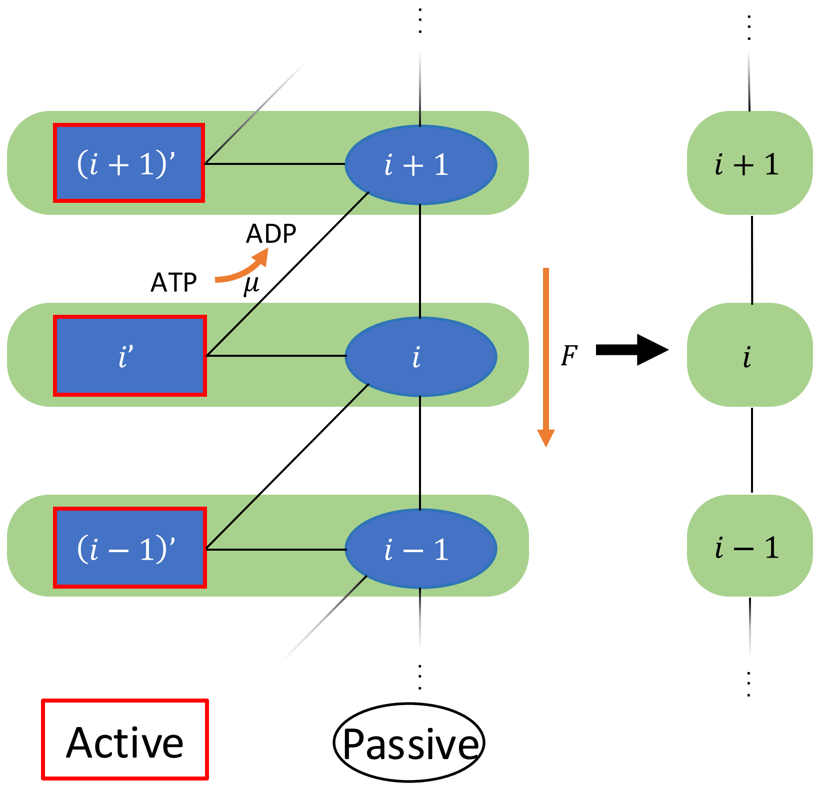
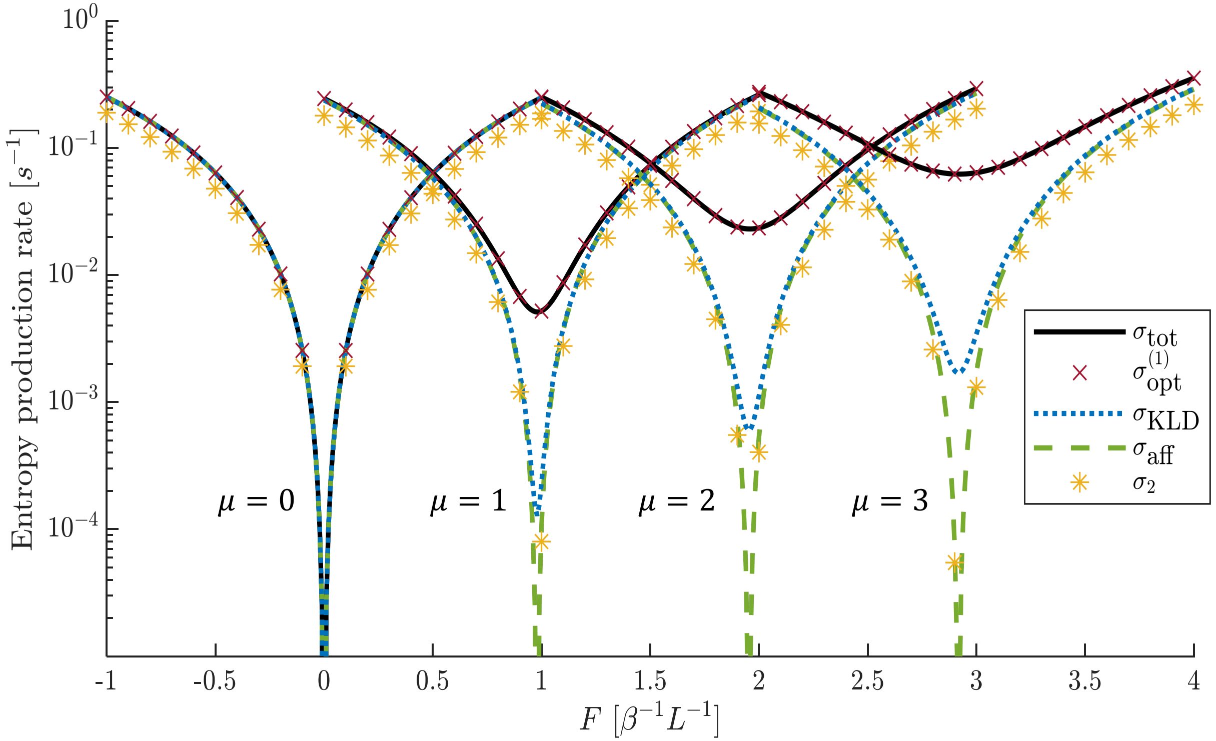
Results

- Molecular motor





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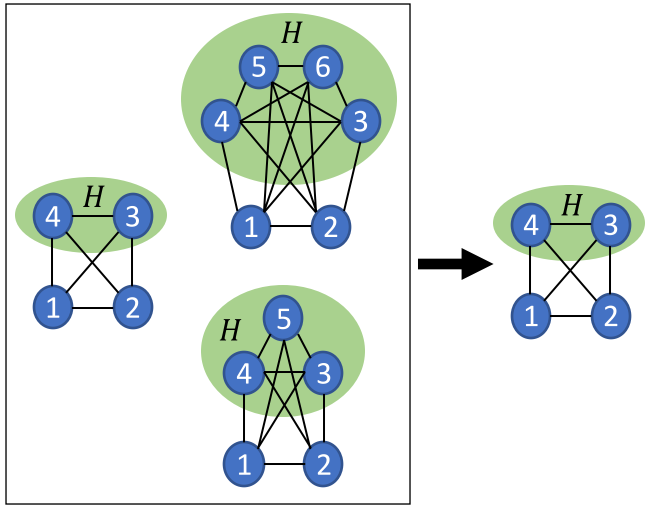
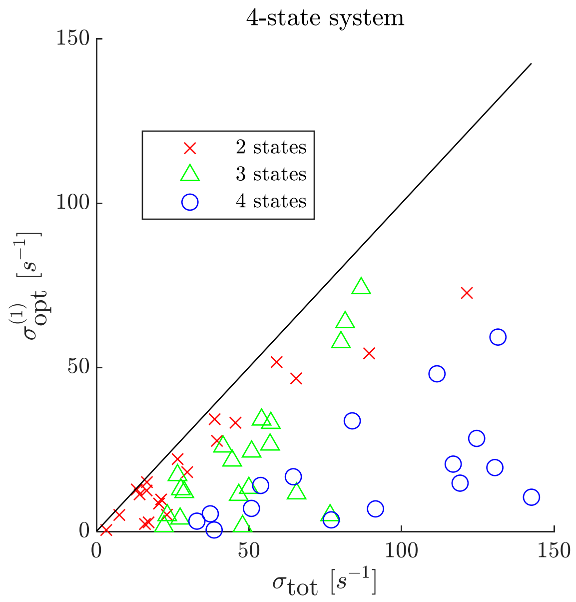
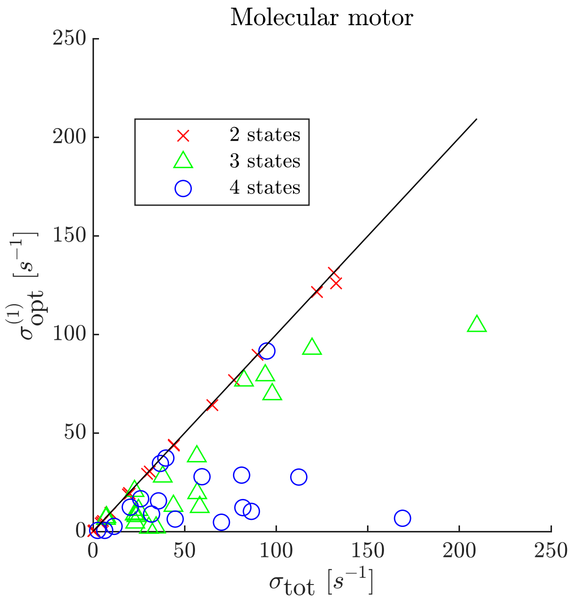
Results

- What if we don't know the underlying topology?
  - ➢We assume a simple model







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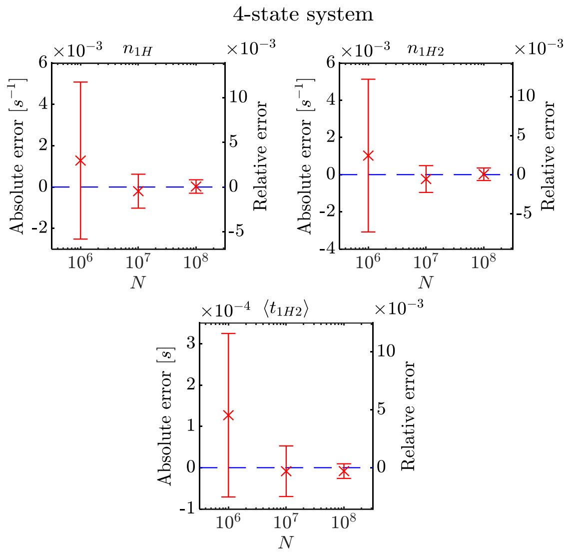
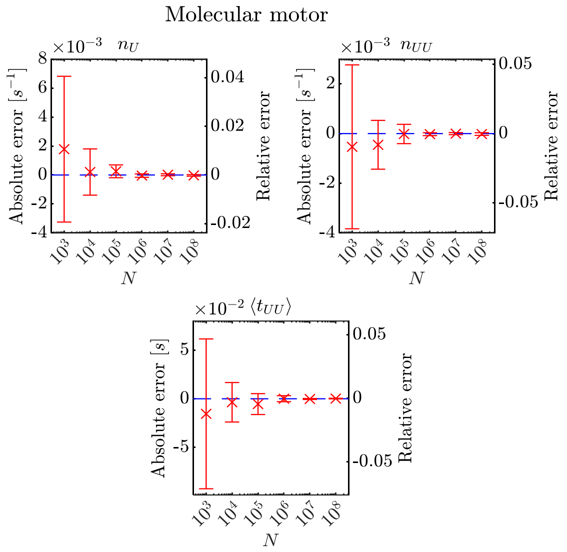
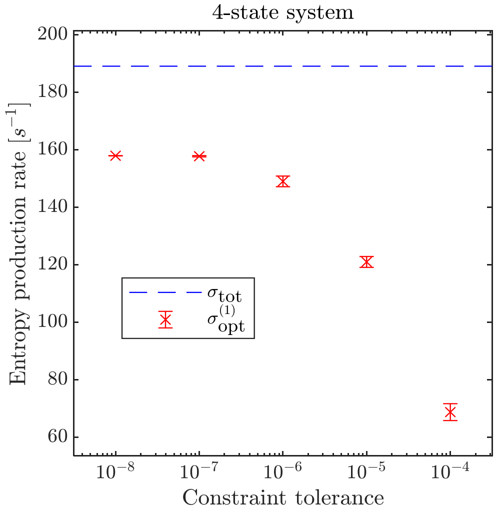
Results

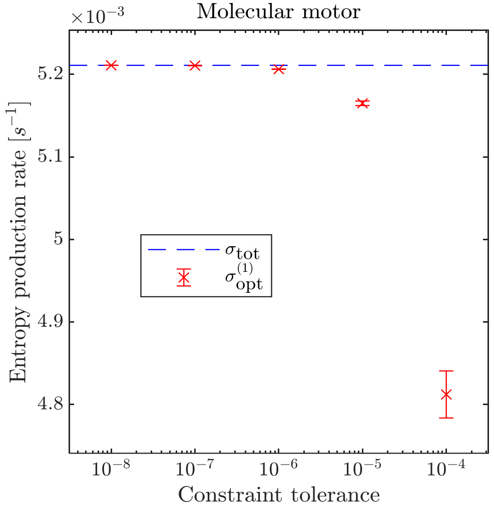
- Accuracy of the statistics





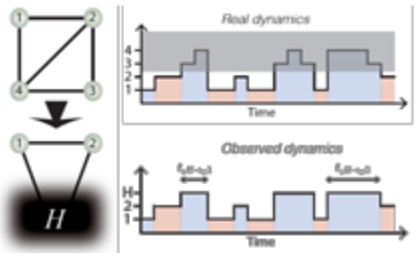
- Constraint tolerance





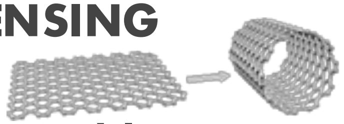
The Bisker Group at Tel Aviv University

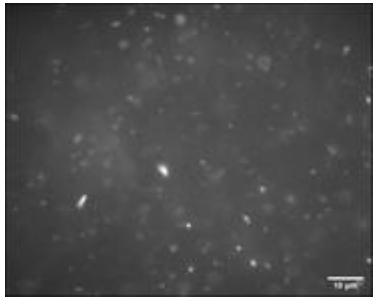
QUANTITATIVE MEASURES OF NONEQUILIBRIUM INFERRED FROM PARTIAL INFORMATION



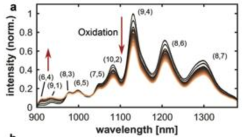
Gili Bisker, et. al., ***J. of Statistical Mechanics***, 2017
Gili Bisker, et. al., ***Nature Comm***. 2019
Aishani Ghosal and Gili Bisker, ***Physical Chemistry Chemical Physics***, 2022
Aishani Ghosal and Gili Bisker, ***Journal of Physics D: Applied Physics***, 2023
Uri Kapustin, Aishani Ghosal, and Gili Bisker, ***Physical Review Research***, 2024
Eden Nitzan, Aishani Ghosal, and Gili Bisker, ***Physical Review Research***, 2023

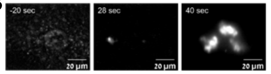
NEAR-INFRARED FLUORESCENT SINGLE-WALLED CARBON NANOTUBES FOR IMAGING AND SENSING

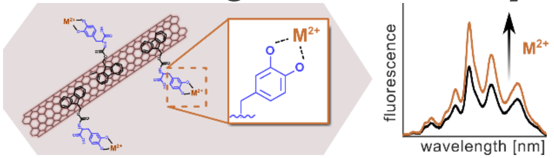




Monitoring self-assembly







Verena Wulf, et. al., ***Advanced Functional Materials***, 2022
Verena Wulf, et. al., ***Nano Letters***, 2022
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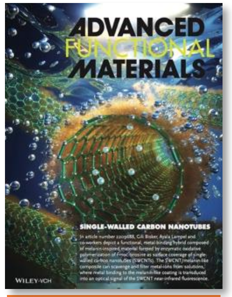
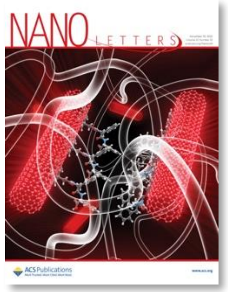
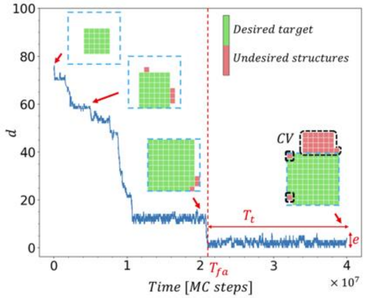




NONEQUILIBRIUM SELF-ASSEMBLY



Gili Bisker and Jeremy L. England, ***PNAS***, 2018
Gili Bisker, ***Nature Physics***, 2020
A Ben-Ari, L Ben-Ari, Gili Bisker, ***The J. of Chem. Phys.***, 2021
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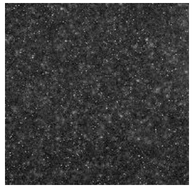
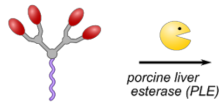
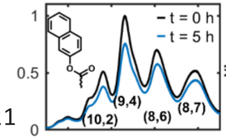
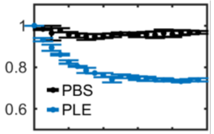
Monitoring enzymatic activity









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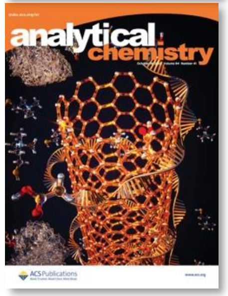

Nanosensors for bioanalytes

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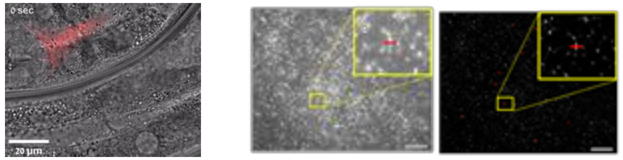
Imaging in the near-IR



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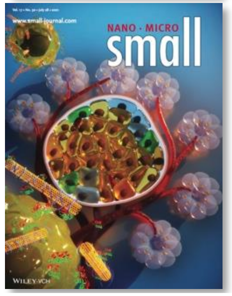

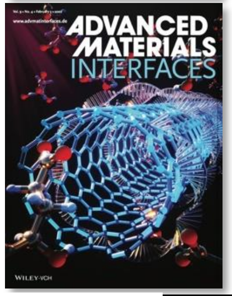

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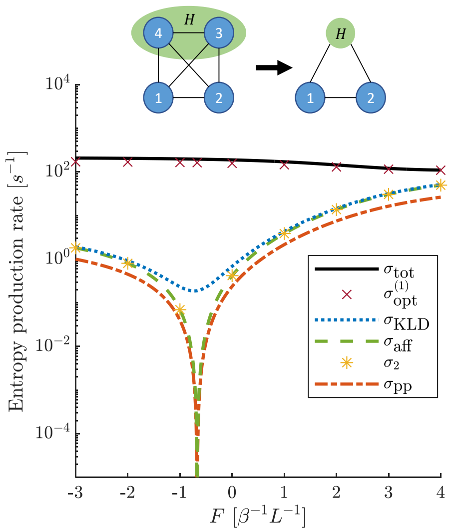
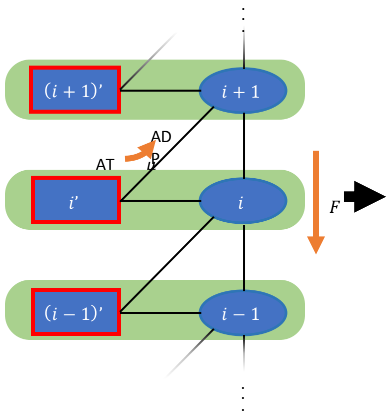
Summary

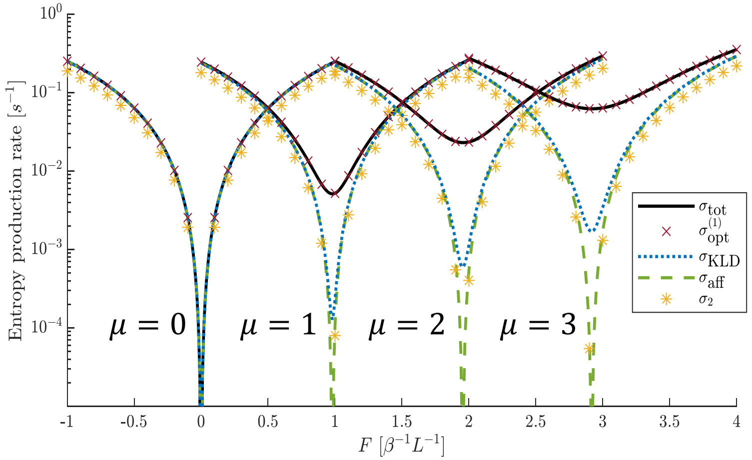
Bound	σ_{pp}	σ_{aff}	σ_{KLD}	σ_2	σ_{opt}
Information used	Transitions between observed Markovian states n_{ij}	Statistics for second-order semi-Markov process $[IJ] \to [JK]$ Transitions only!	Statistics for second-order semi-Markov process $[IJ] \to [JK]$ Transitions and waiting times	First- and second-order transitions statistics of the observed states n_{IJ}, n_{IJK}	Steady states probabilities π_I First- and second-order transitions statistics n_{IJ}, n_{IJK} Waiting time statistics $\langle t_{IJK}^k \rangle$ Underlying topology

<u>Hierarchy of the bounds:</u>

$$\left.\begin{matrix} \sigma_{pp} \le \sigma_{aff} \le \sigma_{KLD} \\ \sigma_{pp} \le \sigma_2 \end{matrix}\right\} \le \sigma_{opt}$$







Other bounds?

<u>**Transition-based statistics**</u>

- Jann Van der Meer, Benjamin Ertel, Udo Seifert, *Physical Review X* 12, 2022
- PE Harunari, A Dutta, M Polettini, É Roldán, *Physical Review X* 12, 2022

<u>**Snippets statistics**</u>

- Jann van der Meer, Julius Degünther, Udo Seifert, *PRL*, 130, 2023

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- AC Barato, U Seifert, PRL, 2015
- TR Gingrich, JM Horowitz, N Perunov, J. England, PRL, 116, 2016

<u>**Milestoning-based estimators**</u>

- Kristian Blom, Kevin Song, Etienne Vouga, Aljaž Godec, and Dmitrii E. Makarov, *PNAS*, 2024

<u>**Reformulation of semi-CG**</u>

Uri Kapustin, Aishani Ghosal, and **Gili Bisker**, *Physical Review Research*, 2024

Thank you

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